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Conformal primaries of $\mathrm{OSp}(8/4, \mathbb{R})$ and BPS states in AdS_4

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Abstract: We derive short UIR's of the $\mathrm{OSp}(8/4, \mathbb{R})$ superalgebra of 3d $N = 8$ superconformal field theories by the requirement that the highest weight states are annihilated by a subset of the super-Poincaré odd generators. We then find a superfield realization of these BPS saturated UIR's as "composite operators" of the two basic ultrashort "supersingleton" multiplets. These representations are the AdS_4 analogue of BPS states preserving different fractions of supersymmetry and are therefore suitable to classify perturbative and non-perturbative excitations of M-theory compactifications.

Keywords: M-Theory, Conformal and W Symmetry, Superspaces.

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Contents

1. Introduction

Superfield representations [1] of super-Poincaré and superconformal algebras have been proved to be useful tools since the early development of supersymmetry for several reasons.

They provide the natural framework to formulate supersymmetric field theories in a "covariant fashion" and allow one, in many cases, to achieve a simple understanding of the softening of "quantum divergences". This milder quantum behaviour of supersymmetric field theories is at the basis of the so-called "non-renormalization theorems" which are one of the striking features of supersymmetric quantum theories [2]. In modern language, which applies to generic supersymmetric theories, these non-renormalization theorems are due to the fact that supersymmetric field theories have some "field representations" that are short, namely, the component field of highest dimension (which is not a total derivative) lies at a lower θ level than what is naively expected from a generic superfield.

Examples of such "short" superfields already appear in $N = 1$ 4d supersymmetry and they are called "chiral" [3]. In the case of superconformal algebras chiral primaries have a "ring structure" under multiplication and their conformal dimension is quantized in terms of the R U(1) charge.

In N-extended supersymmetry in $d = 4$ as well as in other dimensions one needs to generalize the notion of "chiral superfields". The point is that the shortening is often due to an interplay between the conformal dimension and the (non-abelian) R-symmetry quantum numbers. The latter, in $d = 3$ and 6 are related to the Dynkin labels of the SO(N) and USp($2N$) groups while in $d = 4$ for $N \geq 2$, to the Dynkin labels of SU(N).

Extended superspaces, enlarged with coordinates on G/H where G is the R-symmetry of the superconformal algebra and H is a maximal subgroup (with rank of H = rank of G) are called harmonic superspaces [4, 5]. They provide the suitable framework in which the notion of chirality is generalized to Grassmann analyticity [6]. For these "short" superfields the superconformal algebra is realized in a subspace of the full superspace which contains a reduced number of the original anticommuting Grassmann variables.

In the spirit of the AdS/CFT correspondence [7] where boundary "conformal operators" of CFT_d are mapped onto "bulk states" in AdS_{d+1}, multiplet shortening translates into a BPS condition on massive (and massless) particle states in anti-de Sitter space (see, for instance, [8]).

Superconformal algebras in d dimensions appear as vacuum symmetries of string or M-theory compactified on AdS_{d+1}. Massive BPS saturated UIR's of these algebras should therefore be relevant to classify solitons preserving different fractions of supersymmetry, as it happens in the corresponding flat space limit.

The general analysis of multiplet shortening is related to the so-called "unitary bounds" of UIR's of superconformal algebras. For the $d = 4$ case the latter was obtained in the 80's in ref. [9] for $N = 1$ and in ref. [10] for arbitrary N. The relation with the multiplet shortening and the AdS_5/CFT_4 correspondence was recently spelled out in [11].

The superfield analysis in CFT_d is "dual" to the "state" analysis [12]–[15] on AdS_{d+1} since the same superalgebra acts on these representation spaces. However, the superfield approach is more powerful not only because it allows one to treat quantum field theories but because it leads to a simpler classification of "massive representations" in the language of composite operators. The different BPS conditions in AdS_d are rephrased to the different Grassmann analytic operators (generalizations of "chiral operators") which exist in extended harmonic superspace.

The full classification of all BPS conditions was carried out for $d = 4, 6$ superconformal algebras in refs. [16, 17] and it is extended to the $d = 3$ $N = 8$ superconformal algebra in the present paper. The appropriate superconformal algebra is in this case $\mathrm{OSp}(8/4, \mathbb{R})$ which is a different non-compact form of the superalgebra which occurs in the $(2, 0)$ theory in $d = 6$. The latter is related to M-theory on $AdS_7 \times S^4$. The former is appropriate to the $AdS_4 \times S_7$ compactification of M-theory and some of its representations, both massless and massive, have been widely considered in the literature (see, e.g. [15, 18]).

The purpose of this paper is to extend the harmonic superspace analysis to the $d = 3$ $N = 8$ case in order to obtain all BPS states which may occur in AdS_4. These are the AdS analogues of the 1/2, 1/4 and 1/8 BPS states of Poincaré supersymmetry which occur in the classification of extremal black holes in supergravity theories [19, 20]. Therefore BPS states in AdS_4 correspond, in particular, to anti-de Sitter black holes of $N = 8$ gauged SO(8) supergravity [21].

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The paper is organized as follows. In section 2 we carry out a general analysis of the short highest weight UIR's of $\mathrm{OSp}(8/4,\mathbb{R})$. To this end we consider $\mathrm{OSp}(8/4,\mathbb{R})$ as the $N=8$ 3d superconformal algebra and study the conditions on the HWS's which are annihilated by all the S — (conformal supersymmetry) generators and by a fraction (1/2, 3/8, 1/4 or 1/8) of the Q — (Poincaré supersymmetry) ones. As a result we find that the Lorentz spin of these HWS's must vanish and that their conformal dimension should be related to their SO(8) Dynkin labels. Such HWS's generate series of representations exhibiting 1/2, 3/8, 1/4 and 1/8 BPS shortening. The simplest multiplets of maximal shortening (1/2 BPS) are the two distinct "supersingletons". In section 3 and 4 we realize the $N=8$ supersingletons first as constrained superfields in ordinary superspace and then as Grassmann analytic superfields in harmonic superspace. The latter have the advantage that their analyticity properties are preserved by multiplication. This allows us, in section 5, to construct all composite operators obtained by multiplying supersingleton superfields and undergoing different shortenings corresponding to different BPS states in the AdS_4 bulk interpretation. We show that by tensoring only one type of supersingletons we can only construct 1/2, 1/4 and 1/8 BPS states, but by mixing the two types we can reproduce the complete classification of short multiplets from section 2. In this way we also give an indirect proof that all the representations found in section 2 are indeed unitary.

2. Short highest weight UIR's of $\mathrm{OSp}(8/4,\mathbb{R})$

In this section we shall derive the general conditions on the highest weight state (HWS) of a short representation of $\mathrm{OSp}(8/4,\mathbb{R})$.

The superalgebra $\mathrm{OSp}(8/4,\mathbb{R})$ is the $N=8$ superconformal algebra in three dimensions (only the part of the algebra relevant to our argument is shown):

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 2\,\delta^{ij}\,\Gamma^\mu_{\alpha\beta}\,P_\mu\,, \qquad \left\{S^i_\alpha, S^j_\beta\right\} = 2\,\delta^{ij}\,\Gamma^\mu_{\alpha\beta}\,K_\mu\,, \tag{2.1}$$

$$\left\{Q^i_\alpha, S^j_\beta\right\} = \delta^{ij} M_{\alpha\beta} + 2\,\epsilon_{\alpha\beta}\left(T^{ij} + \delta^{ij} D\right), \tag{2.2}$$

$$\left[D, Q^i_\alpha\right] = \frac{i}{2}\,Q^i_\alpha\,, \qquad \left[D, S^i_\alpha\right] = -\frac{i}{2}\,S^i_\alpha\,, \tag{2.3}$$

$$\left[M_{\alpha\beta}, Q^i_\gamma\right] = i\left(\epsilon_{\gamma\alpha}\,Q^i_\beta + \epsilon_{\gamma\beta}\,Q^i_\alpha\right), \qquad \left[M_{\alpha\beta}, S^i_\gamma\right] = i\left(\epsilon_{\gamma\alpha}\,S^i_\beta + \epsilon_{\gamma\beta}\,S^i_\alpha\right), \tag{2.4}$$

$$\left[T^{ij}, Q^k_\alpha\right] = i\left(\delta^{ki}\,Q^j_\alpha - \delta^{kj}\,Q^i_\alpha\right), \qquad \left[T^{ij}, S^k_\alpha\right] = i\left(\delta^{ki}\,S^j_\alpha - \delta^{kj}\,S^i_\alpha\right), \tag{2.5}$$

$$\left[T^{ij}, T^{kl}\right] = i\left(\delta^{ik}\,T^{jl} + \delta^{jl}\,T^{ik} - \delta^{jk}\,T^{il} - \delta^{il}\,T^{jk}\right). \tag{2.6}$$

Here we find the following generators: Q^i_α of $N=8$ Poincaré supersymmetry carrying a 3d spinor Lorentz index $\alpha = 1,2$ and an SO(8) vector[1] index $i = 1,\ldots,8$; S^i_α of

[1] SO(8) has three 8-dimensional representations, 8_v, 8_s and 8_c. Since these three representations

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conformal supersymmetry; P_μ, $\mu = 0,1,2$, of translations; K_μ of conformal boosts; $M_{\alpha\beta} = M_{\beta\alpha}$ of the 3d Lorentz group $\mathrm{SO}(2,1) \sim \mathrm{SL}(2,\mathbb{R})$; D of dilations; $T^{ij} = -T^{ji}$ of SO(8).

The definition of a short representation we adopt requires that its HWS is annihilated by part of the Poincaré supersymmetry generators Q^i_α. Since the latter are irreducible under the Lorentz and R symmetries, the only way to achieve shortening is to break one of them. Postponing the possibility of dealing with the Lorentz group for a future investigation, here we choose to break SO(8) down to $[\mathrm{SO}(2)]^4 \sim [\mathrm{U}(1)]^4$ and decompose the SO(8) vector Q^i_α into eight independent projections carrying different U(1) charges. The first two such projections are:

$$Q^{\pm\pm}_\alpha = \frac{1}{2}\left(Q^1_\alpha \pm Q^2_\alpha\right) \tag{2.7}$$

and the corresponding charge generator is $H_1 = 2iT^{12}$, so that

$$\left[H_1, Q^{\pm\pm}_\alpha\right] = \pm 2\, i\, Q^{\pm\pm}_\alpha\,. \tag{2.8}$$

Note the unusual units of charge, which are spinorial rather than vectorial. Let us write down one of the projections of eq. (2.2) which will be needed in what follows:

$$\left\{Q^{++}_\alpha, S^{--}_\beta\right\} = \frac{1}{2}\, M_{\alpha\beta} + \epsilon_{\alpha\beta}\left(D - \frac{1}{2}\, H_1\right). \tag{2.9}$$

Similarly, we introduce the second charge

$$Q^{(\pm\pm)}_\alpha = \frac{1}{2}\left(Q^3_\alpha \pm Q^4_\alpha\right) \tag{2.10}$$

with generator $H_2 = 2iT^{34}$, so that

$$\left[H_2, Q^{(\pm\pm)}_\alpha\right] = \pm 2\, i\, Q^{(\pm\pm)}_\alpha \tag{2.11}$$

and

$$\left\{Q^{(++)}_\alpha, S^{(--)}_\beta\right\} = \frac{1}{2}\, M_{\alpha\beta} + \epsilon_{\alpha\beta}\left(D - \frac{1}{2}\, H_2\right). \tag{2.12}$$

The third and fourth charges will be introduced in a different way. The components $\underline{i} = 5,6,7,8$ of the 8_v of SO(8) form an SO(4) vector. Since $\mathrm{SO}(4) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)$, we can rewrite it in spinor notation with the help of the Pauli matrices, e.g. $Q^{\underline{i}} \to Q^{\underline{a}\underline{a}'} = Q^{\underline{i}}(\sigma^{\underline{i}})^{\underline{a}\underline{a}'}$. Doing this in eq. (2.2) we obtain

$$\left\{Q^{\underline{a}\underline{a}'}_\alpha, S^{\underline{b}\underline{b}'}_\beta\right\} = \frac{1}{2}\, \epsilon^{\underline{a}\underline{b}}\, \epsilon^{\underline{a}'\underline{b}'} M_{\alpha\beta} - \frac{1}{2}\, \epsilon_{\alpha\beta}\left(t^{\underline{a}\underline{b}}\, \epsilon^{\underline{a}'\underline{b}'} + \epsilon^{\underline{a}\underline{b}}\, t^{\underline{a}'\underline{b}'} - 2\, \epsilon^{\underline{a}\underline{b}}\, \epsilon^{\underline{a}'\underline{b}'} D\right), \tag{2.13}$$

are related by SO(8) triality, the choice which one to ascribe to the supersymmetry generators is purely conventional. In order to be consistent with the other N-extended 3d supersymmetries where the odd generators always belong to the vector representation, we prefer to put an 8_v index i on the supercharges.

where the SU(2) generators t commute with the supersymmetry ones as follows:

$$\left[t^{\underline{ab}}, Q^{\underline{cc'}}\right] = i\left(\epsilon^{\underline{ca}}\, Q^{\underline{bc'}} + \epsilon^{\underline{cb}}\, Q^{\underline{ac'}}\right), \qquad \left[t^{\underline{a'}\,\underline{b'}}, Q^{\underline{cc'}}\right] = i\left(\epsilon^{\underline{c'}\,\underline{a'}}\, Q^{\underline{cb'}} + \epsilon^{\underline{c'}\,\underline{b'}}\, Q^{\underline{ca'}}\right). \tag{2.14}$$

In this notation the two remaining charges are given by

$$H_3 = t^{\underline{12}}, \qquad H_4 = t^{\underline{1'2'}} \tag{2.15}$$

and by denoting $\underline{1} \equiv [+]\,, \underline{2} \equiv [-]$ and $\underline{1'} \equiv \{+\}\,, \underline{2'} \equiv \{-\}$, we find

$$\left[H_3, Q^{[\pm]\{\pm\}}\right] = \left[H_4, Q^{[\pm]\{\pm\}}\right] = \pm i\, Q^{[\pm]\{\pm\}}\,. \tag{2.16}$$

The two relevant projections of eq. (2.2) now are

$$\left\{Q_\alpha^{[+]\{+\}}, S_\beta^{[-]\{-\}}\right\} = \frac{1}{2}\, M_{\alpha\beta} + \epsilon_{\alpha\beta}\left(D - \frac{1}{2}\, H_3 - \frac{1}{2}\, H_4\right), \tag{2.17}$$

$$\left\{Q_\alpha^{[+]\{-\}}, S_\beta^{[-]\{+\}}\right\} = -\frac{1}{2}\, M_{\alpha\beta} - \epsilon_{\alpha\beta}\left(D - \frac{1}{2}\, H_3 + \frac{1}{2}\, H_4\right). \tag{2.18}$$

Besides the four SO(2) charges, the algebra of SO(8) contains $28 - 4 = 24$ generators which can be arranged into 12 "step-up" operators (positive roots):

$$\{\mathcal{T}\}_+ = \begin{cases} T^{++(++)}\,,\ T^{++(--)}\,,\ T^{++[\pm]\{\pm\}}\,; \\ T^{(++)[\pm]\{\pm\}}\,; \\ T^{[++]} \equiv T^{[+]\{+\}[+]\{-\}}\,,\ T^{\{++\}} \equiv T^{[+]\{+\}[-]\{+\}} \end{cases} \tag{2.19}$$

and their complex conjugates (negative roots). Among them only 4 (= rank of SO(8)) are independent, namely, $T^{[++]}$, $T^{\{++\}}$, $T^{++(--)}$, $T^{++[-]\{-\}}$.

Above we have given the decomposition of two of the basic representations of SO(8) under the particular embedding of $[\mathrm{SO}(2)]^4$ that we are using here. These are the 8_v (the supersymmetry generators Q^i) and the adjoint 28 (the SO(8) generators T^{ij}). For future reference we also give the decomposition of the two spinor representations, 8_s (ϕ^a, $a = 1, \ldots, 8$) and 8_c ($\psi^{\dot a}$, $\dot a = 1, \ldots, 8$):

$$\phi^a \longrightarrow \phi^{+(+)[\pm]},\ \phi^{-(-)[\pm]},\ \phi^{+(-)\{\pm\}},\ \phi^{-(+)\{\pm\}}\,, \tag{2.20}$$

$$\sigma^{\dot a} \longrightarrow \sigma^{+(+)\{\pm\}},\ \sigma^{-(-)\{\pm\}},\ \sigma^{+(-)[\pm]},\ \sigma^{-(+)[\pm]}\,. \tag{2.21}$$

This has been obtained by successive reductions: $\mathrm{SO}(8) \to \mathrm{SO}(2) \times \mathrm{SO}(6) \sim \mathrm{U}(1) \times \mathrm{SU}(4) \to [\mathrm{SO}(2)]^2 \times \mathrm{SO}(4) \sim [\mathrm{U}(1)]^2 \times \mathrm{SU}(2) \times \mathrm{SU}(2) \to [\mathrm{SO}(2)]^4 \sim [\mathrm{U}(1)]^4$.

Now we turn to the discussion of the representations of $\mathrm{OSp}(8/4, \mathbb{R})$. Let us denote a generic (quasi primary) superconformal field of the $\mathrm{OSp}(8/4, \mathbb{R})$ algebra by the quantum numbers of its HWS:

$$\mathcal{D}(\ell, J; d_1, d_2, d_3, d_4)\,, \tag{2.22}$$

where ℓ is the conformal dimension, J is the Lorentz spin and d_1, d_2, d_3, d_4 are the Dynkin labels (see, e.g. [22]) of the SO(8) R symmetry. In fact, in our scheme the natural labels are the four charges q_1, q_2, q_3, q_4 (the eigenvalues of $H_1, \ldots, H_4$). So, we can alternatively denote the HWS $|\ell, J, q_i\rangle$. The Dynkin labels $[d_1, d_2, d_3, d_4]$ are related to the U(1) charges (q_1, q_2, q_3, q_4) as follows:

$$d_1 = \frac{1}{2}(q_1 - q_2)\,, \qquad d_2 = \frac{1}{2}(q_2 - q_3 - q_4)\,, \qquad d_3 = q_3\,, \qquad d_4 = q_4\,. \tag{2.23}$$

The above relations can be most easily derived [2] by comparing the Dynkin labels and the charges of the HWS of the following four irreps: $8_v : [1,0,0,0] \leftrightarrow (2,0,0,0)$, $28 : [0,1,0,0] \leftrightarrow (2,2,0,0)$, $8_s : [0,0,1,0] \leftrightarrow (1,1,1,0)$, $8_c : [0,0,0,1] \leftrightarrow (1,1,0,1)$. Note that (2.23) implies restrictions on the allowed values of the charges of a HWS:

$$q_1 - q_2 = 2n \geq 0\,, \quad q_2 - q_3 - q_4 = 2k \geq 0\,, \quad q_3 \geq 0\,, \quad q_4 \geq 0\,. \tag{2.24}$$

A general HWS is defined by a subset of generators of the algebra which annihilate it. These include all the conformal supersymmetry generators:

$$S^i_\alpha|\ell, J, q_i\rangle = 0 \tag{2.25}$$

(and, consequently, the boosts K_μ) as well as the SO(8) "step-up" operators (2.19):

$$\{\mathcal{T}\}_+|\ell, J, q_i\rangle = 0\,. \tag{2.26}$$

The second condition defines $|\ell, J, q_i\rangle$ as the HWS of a UIR of SO(8). A similar condition ensures irreducibility under the Lorentz group. Further, $|\ell, J, q_i\rangle$ should be an eigenstate of the generators D, M^2, H_i fixing its dimension ℓ, spin J and charges q_i.

Now, what makes a multiplet "short" is the additional requirement that part of the supersymmetry charges Q^i_α also annihilate the HWS. When choosing this subset of Q's we have to make sure that it is compatible with the rest of the conditions and with the algebra (2.1)–(2.6). First of all, these Q's must anticommute among themselves, otherwise the first of eqs. (2.1) will yield restrictions on the momentum P_μ. Secondly, eq. (2.26) implies that they must form a closed algebra (a Cauchy-Riemann structure) with all the SO(8) step-up operators $\{\mathcal{T}\}_+$. It is easy to see that such a subset can at most involve four supercharges. In the AdS language such multiplets are called 1/2 BPS ($4 = \frac{1}{2}8$ generators annihilate the HWS). There exist two possible choices:

$$\begin{aligned}\text{type-I } \frac{1}{2}\text{ BPS: } Q^{++}|\ell, J, q_i\rangle &= Q^{(++)}|\ell, J, q_i\rangle = Q^{[+]\{+\}}|\ell, J, q_i\rangle \\ &= Q^{[+]\{-\}}|\ell, J, q_i\rangle = 0\end{aligned} \tag{2.27}$$

[2]We are grateful to L. Castellani for suggesting this to us.

or

$$\text{type-II } \frac{1}{2} \text{ BPS: } Q^{++}|\ell, J, q_i\rangle = Q^{(++)}|\ell, J, q_i\rangle = Q^{[+]\{+\}}|\ell, J, q_i\rangle = Q^{[-]\{+\}}|\ell, J, q_i\rangle = 0\,. \tag{2.28}$$

Finally, conditions (2.27) or (2.28) should be consistent with (2.25). Using the projections (2.9), (2.12), (2.17) and (2.18) of eq. (2.2), we obtain the following constraint on the charges, conformal weight and spin of the HWS:[3]

$$\text{type-I } \frac{1}{2} \text{ BPS: } q_1 = q_2 = q_3 = 2\,\ell\,, \qquad q_4 = 0\,, \qquad J = 0\,; \tag{2.29}$$

$$\text{type-II } \frac{1}{2} \text{ BPS: } q_1 = q_2 = q_4 = 2\,\ell\,, \qquad q_3 = 0\,, \qquad J = 0\,, \tag{2.30}$$

where $2\ell \equiv m$ is a non-negative integer. Computing the Dynkin labels from (2.23), we can say that the 1/2 BPS multiplets above correspond to

$$\text{type-I } \frac{1}{2} \text{ BPS: } \mathcal{D}\left(\frac{m}{2}, 0; 0, 0, m, 0\right); \tag{2.31}$$

$$\text{type-II } \frac{1}{2} \text{ BPS: } \mathcal{D}\left(\frac{m}{2}, 0; 0, 0, 0, m\right). \tag{2.32}$$

Besides the 1/2 BPS conditions there exist weaker shortening conditions. Thus, we can require that a subset of only three supercharges annihilate the HWS. Once again, the choice must be consistent with condition (2.26), and this gives only one possibility:

$$\frac{3}{8} \text{ BPS: } Q^{++}|\ell, J, q_i\rangle = Q^{(++)}|\ell, J, q_i\rangle = Q^{[+]\{+\}}|\ell, J, q_i\rangle = 0\,. \tag{2.33}$$

This is a 3/8 BPS multiplet in the AdS language. This time the condition on the weight, spin and charges is

$$q_1 = q_2 = q_3 + q_4 = 2\,\ell\,, \qquad J = 0\,. \tag{2.34}$$

Denoting $q_3 = m$, $q_4 = n$ where m, n are non-negative integers and computing the Dynkin labels, we find that this type of multiplet corresponds to

$$\frac{3}{8} \text{ BPS: } \mathcal{D}\Big(1/2(m+n), 0; 0, 0, m, n\Big). \tag{2.35}$$

The next step will be to take a subset of two supercharges compatible with (2.26), which is

$$\frac{1}{4} \text{ BPS: } Q^{++}|\ell, J, q_i\rangle = Q^{(++)}|\ell, J, q_i\rangle = 0\,. \tag{2.36}$$

[3] Such relations have been known from the very beginning of supersymmetry, see [23].

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This is a 1/4 BPS multiplet in the AdS language. This time the condition is

$$q_1 = q_2 = 2\,\ell\,, \qquad J = 0\,, \tag{2.37}$$

q_3 and q_4 being only restricted by (2.24). Denoting $q_1 = q_2 = m + n + 2k$, $q_3 = m$, $q_4 = n$ where m, n, k are non-negative integers, we find that this type of multiplet corresponds to

$$\frac{1}{4}\ \text{BPS:}\ \mathcal{D}\Big(1/2(m+n) + k, 0; 0, k, m, n\Big)\,. \tag{2.38}$$

Finally, the weakest shortening condition is obtained by retaining only one supercharge (the HWS among the eight projections of Q^i):

$$\frac{1}{8}\ \text{BPS:}\ Q^{++}|\ell, J, q_i\rangle = 0\,. \tag{2.39}$$

This is a 1/8 BPS multiplet in the AdS language. The condition in this case is

$$q_1 = 2\ell\,, \qquad J = 0\,, \tag{2.40}$$

q_2, q_3 and q_4 satisfying (2.24). Denoting $q_1 = m + n + 2k + 2l$, $q_2 = m + n + 2k$, $q_3 = m$, $q_4 = n$ where m, n, k, l are non-negative integers, we find

$$\frac{1}{8}\ \text{BPS:}\ \mathcal{D}\Big(1/2(m+n) + k + l, 0; l, k, m, n\Big)\,. \tag{2.41}$$

This concludes our abstract analysis of the possible short representations of $\mathrm{OSp}(8/4,\mathbb{R})$. Note that we are not directly addressing the question of whether these representations are unitary or not. However, in the rest of the paper we shall show that all of them can be realized by tensoring two elementary building blocks, the so-called supersingleton representations. Since the latter are known to be UIR's of $\mathrm{OSp}(8/4,\mathbb{R})$, this also answers the above question affirmatively.

3. Supersingletons

Let us consider the simplest $\mathrm{OSp}(8/4,\mathbb{R})$ representations of the type (2.31) or (2.32). They are obtained by setting $m = 1$, so they correspond to $\mathcal{D}(1/2, 0; 0, 0, 1, 0)$ or $\mathcal{D}(1/2, 0; 0, 0, 0, 1)$. Such representations are called "supersingletons" [24, 15]. Each of them is just a collection of 8 Dirac supermultiplets [25] made up of "Di" and "Rac" singletons [26]. We observe that in the framework of the AdS/CFT correspondence [27] the supersingleton describes the microscopic degrees of freedom of an M-2 brane with the scalars being the coordinates transverse to the brane which are then in the 8_v of SO(8). The existence of two distinct types of $N = 8$ 3d supersingletons has first been noted in ref. [28].

Our task now will be to realize the supersingleton in $N = 8$ 3d superspace. Consider first type I. Noting that the HWS in the multiplet $\mathcal{D}(1/2, 0; 0, 0, 1, 0)$ has spin 0 and the Dynkin labels of the 8_s of SO(8), we take a scalar superfield $\Phi_a(x^\mu, \theta_i^\alpha)$ carrying an external 8_s index a.

The superfield Φ_a is a reducible representation of $N = 8$ Poincaré supersymmetry. This can be seen from the fact that the first fermion field in its decomposition,

$$\Phi_a(x^\mu, \theta^\alpha_i) = \phi_a(x) + \theta^\alpha_i\, \psi_{\alpha\, ia}(x) + \cdots\,, \tag{3.1}$$

is reducible under SO(8): $\psi_{\alpha\, ia} \to 8_v \otimes 8_s = 8_c \oplus 56_s$. The way to achieve irreducibility is to impose a constraint [32] on the superfield which removes the 56_s part of $\psi_{\alpha\, ia}$:

$$\text{type I: } D^i_\alpha \Phi_a = \frac{1}{8}\, \gamma^i_{a\dot b}\, \tilde\gamma^j_{\dot b c}\, D^j_\alpha\, \Phi_c\,. \tag{3.2}$$

Here D^i_α are the covariant spinor derivatives satisfying the supersymmetry algebra

$$\left\{D^i_\alpha, D^j_\beta\right\} = 2\, i\, \delta^{ij} (\Gamma^\mu)_{\alpha\beta} \partial_\mu\,. \tag{3.3}$$

The SO(8) gamma matrices $\gamma^i_{a\dot b}$ and $\tilde\gamma^i_{\dot a b} = (\gamma^{iT})_{\dot a b}$ satisfy the Clifford algebra relations

$$\gamma^i_{a\dot b}\, \tilde\gamma^j_{\dot b c} + \gamma^j_{a\dot b}\, \tilde\gamma^i_{\dot b c} = 2\, \delta^{ij}\, \delta_{ac}\,, \qquad \tilde\gamma^i_{\dot a b}\, \gamma^j_{b\dot c} + \tilde\gamma^j_{\dot a b} \gamma^i_{b\dot c} = 2\, \delta^{ij} \delta_{\dot a\dot c}\,. \tag{3.4}$$

Using (3.3) one can show that the constraint (3.2) eliminates all the components of the superfield but two:

$$\begin{aligned}\Phi_a(x^\mu, \theta^\alpha_i) = {} & \phi_a(x) + \theta^\alpha_i (\gamma_i)_{a\dot b}\, \psi_{\alpha\, \dot b}(x) + \theta^\alpha_i\, \theta^\beta_j\, (\gamma_{ij})_{ab}\, i\, \partial_{\alpha\beta}\, \phi_b + \\ & + \theta^\alpha_i\, \theta^\beta_i\, \theta^\gamma_k (\gamma_{ijk})_{a\dot b}\, i\, \partial_{(\alpha\beta} \psi_{\gamma)\dot b} + \theta^\alpha_i\, \theta^\beta_i\, \theta^\gamma_k\, \theta^\delta_l\, (\gamma_{ijkl})_{ab}\, \partial_{(\alpha\beta}\, \partial_{\gamma\delta)}\, \phi_b\,, \end{aligned} \tag{3.5}$$

where $\partial_{\alpha\beta} = \partial_{\beta\alpha} = (\Gamma^\mu)_{\alpha\beta} \partial_\mu$ and $\gamma_{ij\ldots}$ are the anti-symmetrized products of the SO(8) gamma matrices. In addition, the constraint (3.2) puts these fields on shell:

$$\Box \phi_a = 0\,, \qquad \partial^{\alpha\beta} \psi_{\beta\, \dot a} = 0\,. \tag{3.6}$$

Thus, the content of the constrained superfield is a massless multiplet of Poincaré supersymmetry consisting of a scalar in the 8_s and a spinor in the 8_c UIR's of SO(8).[4]

Note that the field equations (3.6) can be obtained from a supersymmetric action [31]. Consequently, the physical fields ϕ_a and $\psi_{\alpha\, \dot a}$ have canonical dimensions 1/2 and 1, respectively. This implies that the superfield Φ_a has dimension 1/2, in accord with the abstract representation $\mathcal{D}(1/2, 0; 0, 0, 1, 0)$.

Finally, the alternative supersingleton representation of type II can be realized in terms of a superfield $\Sigma_{\dot a}$ carrying an 8_c external index and satisfying the constraint

$$\text{type II: } D^i_\alpha\, \Sigma_{\dot a} = \frac{1}{8}\, \tilde\gamma^i_{\dot a b}\, \gamma^j_{b\dot c}\, D^j_\alpha\, \Sigma_{\dot c}\,. \tag{3.7}$$

It describes a massless multiplet consisting of a scalar $\sigma_{\dot a}(x)$ and a spinor $\chi_{\alpha\, a}(x)$ in the 8_c and 8_s, correspondingly.

The problem we want to address now is how to tensor supersingletons. Doing it directly in terms of constrained superfields is quite difficult. Our alternative approach consists in first rewriting the constraints (3.2) or (3.7) as analyticity conditions in harmonic superspace, after which the tensor multiplication becomes straightforward.

[4] Superfield representations of other OSp(N/4) have been considered in the literature [29, 30].

4. The supersingletons as harmonic analytic superfields

The harmonic space suitable for our purposes is given by the coset[5]

$$\frac{\mathrm{SO}(8)}{[\mathrm{SO}(2)]^4} \sim \frac{\mathrm{Spin}(8)}{[\mathrm{U}(1)]^4}\,. \tag{4.1}$$

This is a $28-4=24$-dimensional compact manifold. Instead of trying to introduce explicit coordinates on it, the harmonic method [4] prescribes to use the entire matrices of the fundamental representation of the group to parametrize the coset. The complication in the case of SO(8) is that one has three inequivalent fundamental representations, $8_s, 8_c, 8_v$. The solution to this problem has been found in ref. [35]. One introduces three sets of harmonic variables:

$$u^A_a\,,\, w^{\dot A}_{\dot a}\,,\, v^I_i\,, \tag{4.2}$$

where A, $\dot A$ and I denote the decompositions of an 8_s, 8_c and 8_v index, correspondingly, into sets of four U(1) charges, according to the coset denominator $[\mathrm{U}(1)]^4$ (see section 2 for details). Each of the 8×8 real matrices (4.2) is a matrix of the corresponding representation of SO(8) $\sim$ Spin(8). This implies that all of them are orthogonal matrices (this is a peculiarity of SO(8) due to triality):

$$u^A_a u^B_a = \delta^{AB}\,, \qquad w^{\dot A}_{\dot a} w^{\dot B}_{\dot a} = \delta^{\dot A\dot B}\,, \qquad v^I_i v^J_i = \delta^{IJ} \tag{4.3}$$

(and similarly with small and capital indices interchanged). These matrices supply three copies of the group space (i.e. three sets of 28 real variables each), and we only need one to parametrize the coset (4.1). The condition which identifies the three sets of harmonic variables is

$$u^A_a (\gamma^I)_{A\dot A}\, w^{\dot A}_{\dot a} = v^I_i (\gamma^i)_{a\dot a}\,. \tag{4.4}$$

This relation just expresses the transformation properties of the gamma matrices under SO(8). The reader can convince him(her)self that the conditions (4.3) and (4.4) leave just one set of 28 independent parameters by taking the infinitesimal form of the above matrices. Note that eq. (4.4) can be viewed as the expression of the vector harmonics in terms of the two types of spinor ones. Therefore we shall choose u, w as our harmonic variables.[6]

[5] A formulation of the above multiplet in harmonic superspace has been proposed in ref. [32] (see also [33] and [34] for a general discussion of 3-dimensional harmonic superspaces). The harmonic coset used in [32] is Spin(8)/U(4). Although the supersingleton itself does indeed live on this smaller coset, the residual symmetry U(4) will turn out too big when we start tensoring different realizations of the supersingleton. For this reason we prefer from the very beginning to use the coset (4.1) with a minimal residual symmetry (see also [11] for a discussion of this point).

[6] Although each of the three sets of harmonic variables depends on the same 28 parameters, we need at least two sets to be able to reproduce all possible representations of SO(8).

The idea of the harmonic description of the coset (4.1) is to consider harmonic functions defined as functions of the above sets of variables modulo transformations of $[\mathrm{U}(1)]^4$. In other words, a harmonic function always carries a set of four U(1) charges. These functions are then given by their "harmonic expansions" in terms of all the products of harmonic variables having the same charges. Take, for instance, the function

$$\phi^{+(+)[+]}(u,w) = \phi_a\, u_a^{+(+)[+]} + \phi_{abc}\, u_a^{+(+)[+]}\, u_b^{+(+)[+]}\, u_c^{-(-)[-]} + \\ + \phi_{a\dot b\dot c}\, u_a^{+(+)[+]}\, w_{\dot b}^{+(+)\{+\}}\, w_{\dot c}^{-(-)\{-\}} + \cdots . \tag{4.5}$$

Although the harmonic function only transforms under $[\mathrm{U}(1)]^4$, the coefficients in its expansion are representations of SO(8) $\sim$ Spin(8). Thus, a harmonic function is a collection of an infinite set of irreps of SO(8).

In order to make the harmonic functions irreducible we have to impose differential constraints on them. To this end we introduce harmonic derivatives (the covariant derivatives on the coset (4.1)):

$$D^{IJ} = u_a^A \left(\gamma^{IJ}\right)^{AB} \frac{\partial}{\partial u_a^B} + w_{\dot a}^{\dot A} \left(\gamma^{IJ}\right)^{\dot A\dot B} \frac{\partial}{\partial w_{\dot a}^{\dot B}} + v_i^{[I} \frac{\partial}{\partial v_i^{J]}} . \tag{4.6}$$

They respect the algebraic relations (4.3) and (4.4) among the harmonic variables. Moreover, these derivatives form the algebra of SO(8) realized on the $[\mathrm{U}(1)]^4$ projected indices $A, \dot A, I$ of the harmonics. Four of them just count the four U(1) charges, i.e. the harmonic functions are their eigenfunctions:

$$H_n\, f^{(q_1,q_2,q_3,q_4)}(u,w) = q_n\, f^{(q_1,q_2,q_3,q_4)}(u,w)\,, \quad n = 1,2,3,4\,. \tag{4.7}$$

The remaining 24 ones are the true covariant derivatives on the coset. In our complex $[\mathrm{U}(1)]^4$ notation these are

$$\{\mathcal{D}\}_+ = \begin{cases} D^{++(++)}\,,\ D^{++(--)}\,,\ D^{++[\pm]\{\pm\}}\,; \\ D^{(++)[\pm]\{\pm\}}\,; \\ D^{[++]} \equiv D^{[+]\{+\}[+]\{-\}}\,,\ D^{\{++\}} \equiv D^{[+]\{+\}[-]\{+\}} \end{cases} \tag{4.8}$$

and their complex conjugates. It is clear that the 12 derivatives (4.8) correspond to the step-up operators of SO(8), see (2.19). Therefore we can make a harmonic function irreducible by demanding that all of the derivatives (4.8) annihilate it. In other words, this differential condition reduces the harmonic function to a polynomial corresponding to a highest weight of an SO(8) irrep. For example, the constraint

$$\{\mathcal{D}\}_+ \phi^{+(+)[+]}(u,w) = 0 \Longrightarrow \phi^{+(+)[+]}(u,w) = \phi_a u_a^{+(+)[+]} \tag{4.9}$$

reduces the function (4.5) to an 8_s. This can easily be generalized to any function of the type (4.7) satisfying the constraint

$$\{\mathcal{D}\}_+ f^{(q_1,q_2,q_3,q_4)}(u,w) = 0\,. \tag{4.10}$$

This is the defining condition of the HWS of a UIR of SO(8) given by the Dynkin labels from eq. (2.23). The function satisfying (4.10) is thus reduced to a polynomial of the harmonic variables:

$$\begin{aligned} f^{(q_1,q_2,q_3,q_4)}(u,w) &= f^{(2d_1+2d_2+d_3+d_4,2d_2+d_3+d_4,d_3,d_4)}(u,w) \\ &= f_{a\cdots b\cdots c\cdots \dot{d}\cdots}\left(u_a^{+(+)[+]}\right)^{d_2+d_3}\left(u_b^{+(+)[-]}\right)^{d_2}\times \\ &\quad\times\left(u_c^{+(-)\{-\}}\right)^{d_1}\left(w_{\dot{d}}^{+(+)\{+\}}\right)^{d_1+d_4}. \end{aligned} \tag{4.11}$$

Concluding the discussion of the harmonic coset (4.1) we can say that if one introduces complex coordinates on it, the conditions (4.10) take the form of (covariant) analyticity conditions. For this reason we can call eq. (4.10) "harmonic analyticity" conditions.

The purpose of introducing harmonic variables is to be able to project the super-singleton defining constraint (3.2) (or (3.7)) in an SO(8) covariant way. This means to convert the indices i and a into U(1) charges with the help of the corresponding harmonics: $D^i_\alpha \to D^I_\alpha = v^I_i D^i_\alpha$ and $\Phi_a \to \Phi^A = u^A_a \Phi_a$. Then, using the relation (4.4) it is easy to show that, e.g. the projection $\Phi^{+(+)[+]}$ satisfies the following constraints:

$$D^{++}\,\Phi^{+(+)[+]} = D^{(++)}\,\Phi^{+(+)[+]} = D^{[+]\{\pm\}}\,\Phi^{+(+)[+]} = 0\,. \tag{4.12}$$

We see that half of the spinor derivatives annihilate the superfield $\Phi^{+(+)[+]}$. This is the superspace realization of the 1/2 BPS shortening condition (2.27). Since these spinor derivatives anticommute among themselves (as follows from (3.3) after the appropriate projections), there exists a basis in superspace where $\Phi^{+(+)[+]}$ becomes just a function of half of the odd variables as well as of the harmonic variables:

$$\text{type I: } \Phi^{+(+)[+]} = \Phi^{+(+)[+]}\Big(x_A, \theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, u, w\Big)\,, \tag{4.13}$$

where

$$x_{A\alpha\beta} = x_{\alpha\beta} + i\,\theta^{++}_{(\alpha}\,\theta^{--}_{\beta)} + i\,\theta^{(++)}_{(\alpha}\,\theta^{(--)}_{\beta)} + i\,\theta^{[+]\{+\}}_{(\alpha}\,\theta^{[-]\{-\}}_{\beta)} + i\,\theta^{[+]\{-\}}_{(\alpha}\,\theta^{[-]\{+\}}_{\beta)}\,. \tag{4.14}$$

We can say that $\Phi^{+(+)[+]}$ is a "Grassmann analytic"[7] or a "short" superfield.

So far eqs. (4.12) have been derived as a corollary of the defining constraint (3.2). In order to make the latter equivalent to the former we have to eliminate the harmonic dependence in the superfield (4.13). This is done by imposing another set of constraints, namely, the harmonic analyticity conditions (4.10):

$$\{\mathcal{D}\}_+\Phi^{+(+)[+]}\Big(x, \theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, u, w\Big) = 0\,. \tag{4.15}$$

[7]Grassmann analyticity [4] (see also [5] for a further generalization) is the analog of the notion of chirality [3] from $N=1$ $d=4$ supersymmetry in the case of extended supersymmetry.

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Note that these new constraints are compatible with (4.12) since the two sets of derivatives form a closed algebra (a Cauchy-Riemann structure in the terminology of ref. [36]). It should be stressed that eq. (4.15) now has implications other than just restricting the harmonic dependence. The reason is that in the superspace basis (4.14) where Grassmann analyticity becomes manifest some of the harmonic derivatives from the set $\{\mathcal{D}\}_+$ acquire torsion terms, e.g. $D^{++(++)} = \partial^{++(++)}_{u,w} + i\theta^{++}\Gamma^\mu\theta^{(++)}\partial_\mu$, $D^{++[+]\{\pm\}} = \partial^{++[+]\{\pm\}}_{u,w} + i\theta^{++}\Gamma^\mu\theta^{[+]\{\pm\}}\partial_\mu$, etc. This yields space-time derivative constraints on the components of the superfield $\Phi^{+(+)[+]}$. All this amounts to $\Phi^{+(+)[+]}$ becoming "ultrashort":

$$\begin{aligned}\Phi^{+(+)[+]} &= u^{+(+)[+]}_a\phi_a(x) + \\ &\quad + \Big(\theta^{[+]\{-\}\alpha}w^{+(+)\{+\}}_{\dot a} - \theta^{[+]\{+\}\alpha}w^{+(+)\{-\}}_{\dot a} - \\ &\qquad - \theta^{++\alpha}w^{-(+)[+]}_{\dot a} - \theta^{(++)\alpha}w^{+(-)[+]}_{\dot a}\Big)\psi_{\dot a\,\alpha}(x) + \text{ derivative terms}\,,\end{aligned} \tag{4.16}$$

where the fields are massless. In this way we recover the content (3.5) and (3.6) of the ordinary constrained superfield describing the supersingleton multiplet.

It is instructive to comment on the structure of the two terms in eq. (4.16). The first one is the component at level 0 in the θ expansion. It is a harmonic function of the type (4.9), i.e. a harmonic-projected 8_s. The situation at level 1 is more complicated. Originally, one finds a collection of spinor fields with a variety of charges. In order to find out which one among them is the HWS of an SO(8) representation, we have to look at the accompanying θ's. It is easy to see that $\theta^{[+]\{-\}}$ can serve as a starting point for obtaining the rest by successive applications of the harmonic derivatives $\{\mathcal{D}\}_+$ (the step-up operators of SO(8)):

$$\theta^{[+]\{-\}} \xrightarrow{D^{\{++\}}} \theta^{[+]\{+\}} \xrightarrow{D^{(++)[-]\{-\}}} \theta^{(++)} \xrightarrow{D^{++(--)}} \theta^{++}\,. \tag{4.17}$$

At the same time, $\theta^{[+]\{-\}}$ cannot be obtained from any other of the projections available in the Grassmann analytic superspace. As a consequence, the harmonic analyticity condition (4.15) mixes up the corresponding spinor fields (coefficients at level 1 in the θ expansion), with the exception of the one in the term $\theta^{[+]\{-\}\alpha}\psi^{+(+)\{+\}}_\alpha(x,u,w)$. The latter must satisfy the condition $\{\mathcal{D}\}_+\psi^{+(+)\{+\}}_\alpha = 0$. This means that we are dealing with the HWS of the representation $(1,1,0,1) \leftrightarrow [0,0,0,1]$, i.e. with an 8_c. The remaining level 1 coefficients are related to this HWS by harmonic equations like, e.g. $D^{\{++\}}\psi^{+(+)\{-\}}_\alpha = \psi^{+(+)\{+\}}_\alpha$, etc. In other words, they correspond to different projections ("lower weights") of this 8_c.

The same argument explains why there are no new fields beyond level 1. Indeed, among all the level 2 θ structures we find two which cannot be obtained by acting with the step-up operators on any other structure:

$$\theta^{[+]\{-\}\alpha}\,\theta^{[+]\{-\}}_\alpha\,A^{(1,1,-1,2)}\,, \qquad \theta^{[+]\{-\}\alpha}\,\theta^{[+]\{+\}\beta}\,B^{(1,1,-1,0)}_{(\alpha\beta)} \tag{4.18}$$

corresponding to a scalar and a vector fields. Now, harmonic analyticity again implies that these fields should be highest weights of SO(8) irreps, but their charges do not satisfy the restrictions (2.24). The conclusion is that there are no such independent fields in the expansion of the analytic superfield $\Phi^{+(+)[+]}$ (more precisely, $A^{(1,1,-1,2)} = 0$ and $B^{(1,1,-1,0)}_{(\alpha\beta)} = i\,\partial_{\alpha\beta}\,\phi_a\,u_a^{+(+)[-]}$; such terms are denoted as "derivative terms" in (4.16)).

In conclusion we note that the alternative form of the supersingleton (3.7) is described by the superfield

$$\text{type II: } \Sigma^{+(+)\{+\}}\left(\theta^{++},\theta^{(++)},\theta^{[\pm]\{+\}}\right) \tag{4.19}$$

satisfying the same harmonic constraints (4.15) but depending on a different set of four odd variables. Also, the charges and Dynkin labels of the first component are those of an 8_c instead of 8_s. This is the superspace realization of the 1/2 BPS shortening condition (2.28).

5. Short multiplets as supersingleton "composite operators"

In the preceding section, with the help of the harmonic variables, we have been able to equivalently rewrite the supersingleton as an ultrashort superfield satisfying both conditions of Grassmann (eq. (4.12) or eq. (4.19)) and harmonic (eq. (4.15)) analyticity. The main advantage of this new analytic form of the supersingleton is the possibility to tensor copies of it in a straightforward way and thus to obtain series of short composite multiplets. As we shall show in this section, this procedure allows us to realize all the abstract short $OSp(8/4,\mathbb{R})$ multiplets of section 2.

We observe that in the AdS/CFT correspondence the supersingleton multiplet describing the dynamics of many M-2 branes is endowed with an internal symmetry index and composite operators are further restricted to be singlets under the invariance group [37].

The simplest example of a tensor product is obtained by taking p identical copies of type-I supersingletons, $(\Phi^{+(+)[+]})^p$. Clearly, it satisfies the same constraints of Grassmann and harmonic analyticity. However, the latter is not as strong as before. The reason is that the external charges of the superfield have changed, and the consequences of harmonic analyticity strongly depend on the charges, as the argument at the end of the preceding section has shown. So, for generic $p \geq 4$ the θ expansion goes up to the maximal level 8:

$$\begin{aligned}\left(\Phi^{+(+)[+]}\right)^p = {}& \phi^{[0,0,p,0]} + \theta^{[+]\{-\}\alpha}\,\psi_\alpha^{[0,0,p-1,1]} + \cdots + \\ & + \left(\theta^{[+]\{-\}}\right)^2 A^{[0,0,p-2,2]} + \cdots + \theta^{[+]\{-\}\alpha}\,\theta^{[+]\{+\}\beta} B^{[0,1,p-2,0]}_{(\alpha\beta)} + \cdots + \\ & + \left(\theta^{[+]\{-\}}\right)^2 \theta^{[+]\{+\}\alpha}\,\chi_\alpha^{[0,1,p-3,1]} + \cdots +\end{aligned}$$

$$
\begin{aligned}
&+\theta^{[+]\{-\}\alpha}\,\theta^{[+]\{+\}\beta}\,\theta^{(++)\gamma}\,\rho^{[1,0,p-2,0]}_{(\alpha\beta\gamma)}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\left(\theta^{[+]\{+\}}\right)^2 C^{[0,2,p-4,0]}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\theta^{[+]\{+\}\alpha}\,\theta^{(++)\beta}D^{[1,0,p-3,1]}_{(\alpha\beta)}+\cdots+\\
&+\theta^{[+]\{-\}\alpha}\,\theta^{[+]\{+\}\beta}\,\theta^{(++)\gamma}\,\theta^{++\delta}E^{[0,0,p-2,0]}_{(\alpha\beta\gamma\delta)}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\left(\theta^{[+]\{+\}}\right)^2\theta^{(++)\alpha}\,\sigma^{[1,1,p-4,0]}_{\alpha}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\theta^{[+]\{+\}\alpha}\,\theta^{(++)\beta}\,\theta^{++\gamma}\omega^{[0,0,p-3,1]}_{(\alpha\beta\gamma)}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\left(\theta^{[+]\{+\}}\right)^2\left(\theta^{(++)}\right)^2F^{[2,0,p-4,0]}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\left(\theta^{[+]\{+\}}\right)^2\theta^{(++)\alpha}\theta^{++\beta}G^{[0,1,p-4,0]}_{(\alpha\beta)}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\left(\theta^{[+]\{+\}}\right)^2\left(\theta^{(++)}\right)^2\theta^{++\alpha}\,\tau^{[1,0,p-4,0]}_{\alpha}+\cdots+\\
&+\left(\theta^{[+]\{-\}}\right)^2\left(\theta^{[+]\{+\}}\right)^2\left(\theta^{(++)}\right)^2\left(\theta^{++}\right)^2H^{[0,0,p-4,0]}+\cdots+\\
&+\text{derivative terms}\,. \qquad (5.1)
\end{aligned}
$$

Here we have shown only the leading term at each level and of each Lorentz structure. This is the term whose coefficient is the HWS of an SO(8) irrep. The other terms of the same type contain different harmonic projections of the same component field. Further, instead of the charges we have directly indicated the corresponding Dynkin labels of each component field. Note that the level in the expansion also determines the conformal dimension of the components (given the fact that the dimension of the first component is $p/2$ and that of a θ is $-1/2$).

We see that $(\Phi^{+(+)[+]})^p$ is a short superfield (it depends on half of the odd variables) of the type (2.31), but not an ultrashort one, unlike the supersingleton itself (the case $p=1$). Still, for $p=2,3$ certain terms in the expansion (5.1) are absent if conditions (2.24) are not satisfied. In addition, for $p=2$ one finds conservation conditions for the fields of spins 2, 3/2 and 1, $\partial^{\alpha\beta}E^{[0,0,0,0]}_{(\alpha\beta\gamma\delta)}=\partial^{\alpha\beta}\rho^{[1,0,0,0]}_{(\alpha\beta\gamma)}=\partial^{\alpha\beta}B^{[0,1,0,0]}_{(\alpha\beta)}=0$. This is most easily seen for the top spin 2 which is the only SO(8) singlet in the expansion and hence its divergence cannot be matched by any other component.

The expansion (5.1) reproduces (up to triality) the content of the short multiplets of $\mathrm{OSp}(8/4,\mathbb{R})$ found in refs. [14, 15].

Further short multiplets can be obtained by tensoring different analytic superfields describing the type-I supersingleton. The point is that in section 4 we chose a particular projection of the defining constraint (3.2) which lead to the analytic superfield $\Phi^{+(+)[+]}$. In fact, we could have done this in a variety of ways, each time obtaining superfields depending on different halves of the total number of odd variables. If we decide to always leave out the lowest weight θ^{--} in the 8_v formed by the θ's, we can have four (as many as the rank of SO(8)) distinct but equivalent analytic descriptions of the type-I supersingleton:

$$\begin{aligned}
&\Phi^{+(+)[+]}\left(\theta^{++},\theta^{(++)},\theta^{[+]\{+\}},\theta^{[+]\{-\}}\right),\\
&\Phi^{+(+)[-]}\left(\theta^{++},\theta^{(++)},\theta^{[-]\{+\}},\theta^{[-]\{-\}}\right),\\
&\Phi^{+(-)\{+\}}\left(\theta^{++},\theta^{(--)},\theta^{[+]\{+\}},\theta^{[-]\{+\}}\right),\\
&\Phi^{+(-)\{-\}}\left(\theta^{++},\theta^{(--)},\theta^{[+]\{-\}},\theta^{[-]\{-\}}\right). \qquad (5.2)
\end{aligned}$$

Then we can tensor them in the following way:

$$\begin{aligned}
&\left(\Phi^{+(+)[+]}\right)^{p+q+r+s}\left(\Phi^{+(+)[-]}\right)^{q+r+s}\left(\Phi^{+(-)\{+\}}\right)^{r+s}\left(\Phi^{+(-)\{-\}}\right)^{s} = \phi^{[r+2s,q,p,r]}+\cdots+\\
&\quad+\theta^{[+]\{-\}}_{\alpha_1}\,\theta^{[+]\{+\}}_{\alpha_2}\,\theta^{(++)}_{\alpha_3}\,\theta^{++}_{\alpha_4}A^{[r+2s,q,p-2,r](\alpha_1\cdots\alpha_4)}+\cdots+\\
&\quad+\theta^{[+]\{-\}}_{\alpha_1}\,\theta^{[+]\{+\}}_{\alpha_2}\,\theta^{(++)}_{\alpha_3}\,\theta^{++}_{\alpha_4}\,\theta^{[-]\{+\}}_{\alpha_5}\,\theta^{[-]\{-\}}_{\alpha_6}B^{[r+2s,q-1,p,r](\alpha_1\cdots\alpha_6)}+\cdots+\\
&\quad+\theta^{[+]\{-\}}_{\alpha_1}\,\theta^{[+]\{+\}}_{\alpha_2}\,\theta^{(++)}_{\alpha_3}\,\theta^{++}_{\alpha_4}\,\theta^{[-]\{+\}}_{\alpha_5}\,\theta^{[-]\{-\}}_{\alpha_6}\,\theta^{(--)}_{\alpha_7}\,\chi^{[r+2s-1,q,p,r](\alpha_1\cdots\alpha_7)}+\cdots. \qquad (5.3)
\end{aligned}$$

Here we have shown the first component which belongs to the SO(8) UIR $[r+2s,q,p,r]$ and has conformal dimension $\ell=\frac{1}{2}(p+2q+3r+4s)$ (this follows from the fact that the basic supersingleton has dimension 1/2). In (5.3) one can also see the top spin of each particular series: $J_{\rm top}=2$ if $q=r=s=0$, $J_{\rm top}=3$ if $r=s=0$ or $J_{\rm top}=7/2$ if either $r\neq 0$ or $s\neq 0$. The dimension of the top spin is $\ell[J_{\rm top}]=\frac{1}{2}(p+2q+3r+4s)+J_{\rm top}$ (since each θ carries dimension $-1/2$). Note the absence of a series with top spin $J=5/2$: the reason is that the tensor product of the different realizations (5.2) of the type-I supersingleton can depend on 4, 6 or 7 θ's but not on 5.

The above result can be summarized as follows. By considering composite operators made out of type-I supersingletons we have constructed the following series of $OSp(8/4,\mathbb{R})$ UIR's exhibiting 1/8, 1/4 or 1/2 BPS shortening:

$$\begin{aligned}
&\frac{1}{8}\ \text{BPS:}\ \ \mathcal{D}\left(d_1+d_2+\frac{1}{2}(d_3+d_4),0;d_1,d_2,d_3,d_4\right), \qquad d_1-d_4=2s\geq 0\,;\\
&\frac{1}{4}\ \text{BPS:}\ \ \mathcal{D}\left(d_2+\frac{1}{2}d_3,0;0,d_2,d_3,0\right);\\
&\frac{1}{2}\ \text{BPS:}\ \ \mathcal{D}\left(\frac{1}{2}d_3,0;0,0,d_3,0\right). \qquad (5.4)
\end{aligned}$$

We see that tensoring only one type of supersingletons cannot reproduce the general result of section 2 for all possible short multiplets. Most notably, in (5.4) there is no 3/8 series. The latter can be obtained by mixing the two types of supersingletons:

$$\left[\Phi^{+(+)[+]}\left(\theta^{++},\theta^{(++)},\theta^{[+]\{\pm\}}\right)\right]^{p+q}\left[\Sigma^{+(+)\{+\}}\left(\theta^{++},\theta^{(++)},\theta^{[\pm]\{+\}}\right)\right]^{q}, \qquad (5.5)$$

or the same with Φ and Σ exchanged. Counting the charges and the dimension, we find exact matching with the series (2.35). Further, mixing two realizations of type-I and one of type-II supersingletons, we can construct the 1/4 series

$$\left[\Phi^{+(+)[+]}\right]^{m+k}\left[\Phi^{+(+)[-]}\right]^{k}\left[\Sigma^{+(+)\{+\}}\right]^{n} \tag{5.6}$$

which corresponds to (2.38). Finally, the full 1/8 series (2.41) (i.e. without the restriction $d_1 - d_4 = 2s$ in (5.4)) can be obtained in a variety of ways.

6. Conclusions

In this paper we have analyzed all short highest weight UIR's of the OSp$(8/4, \mathbb{R})$ superalgebra whose HWS's are annihilated by part of the super-Poincaré odd generators. In the field theory language, highest weight reps correspond to conformal quasi primary superfields. Short reps correspond to superfields which do not depend on some of the odd coordinates, a concept generalizing the notion of chiral superfields of $N = 1$ 4d field theories. The number of distinct possibilities have been shown to correspond to different BPS conditions on the HWS. When the algebra is interpreted on the AdS_4 bulk, for which the 3d superconformal field theory corresponds to the boundary M-2 brane dynamics, these states appear as BPS massive excitations, such as K-K states or AdS black holes, of M-theory on $AdS_4 \times S^7$. Since in M-theory there is only one type of supersingleton related to the M-2 brane transverse coordinates [38], according to our analysis massive states cannot be 3/8 BPS saturated, exactly as it happens in M-theory on $M^4 \times T^7$. Indeed, the missing solution was also noticed in ref. [39] by studying AdS_4 black holes in gauged $N = 8$ supergravity. Curiously, in the ungauged theory, which is in some sense the flat limit of the former, the 3/8 BPS states are forbidden [20] by the underlying $E_{7(7)}$ symmetry of $N = 8$ supergravity [40].

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